Polar, Parametric & Vector Review

CHAPTER REVIEW EXERCISES

1. Which of the following curves pass through the point $(1, 4)$?

(a) $c(t) = (t^2, t+3)$ (b) $c(t) = (t^2, t-3)$

(c) $c(t) = (t^2, 3-t)$ (d) $c(t) = (t-3, t^2)$

2. Find parametric equations for the line through $P = (2, 5)$ perpendicular to the line $y = 4x - 3$.

3. Find parametric equations for the circle of radius 2 with center $(1, 1)$. Use the equations to find the points of intersection of the circle with the x- and y-axes.

4. Find a parametrization $c(t)$ of the line $y = 5 - 2x$ such that $c(0) = (2, 1)$.

5. Find a parametrization $c(\theta)$ of the unit circle such that $c(0) = (-1, 0)$.

6. Find a path $c(t)$ that traces the parabolic arc $y = x^2$ from $(0, 0)$ to $(3, 9)$ for $0 \le t \le 1$.

7. Find a path $c(t)$ that traces the line $y = 2x + 1$ from $(1, 3)$ to $(3, 7)$ for $0 \le t \le 1$.

8. Sketch the graph $c(t) = (1 + \cos t, \sin 2t)$ for $0 \le t \le 2\pi$ and draw arrows specifying the direction of motion.

In Exercises 9–12, express the parametric curve in the form $y = f(x)$.

9. $c(t) = (4t - 3, 10 - t)$

10. $c(t) = (t^3 + 1, t^2 - 4)$

11. $c(t) = \left(3 - \frac{2}{t}, t^3 + \frac{1}{t}\right)$

12. $x = \tan t, \quad y = \sec t$

In Exercises 13–16, calculate dy/dx at the point indicated.

13. $c(t) = (t^3 + t, t^2 - 1), \quad t = 3$

14. $c(\theta) = (\tan^2\theta, \cos\theta), \quad \theta = \frac{\pi}{4}$

15. $c(t) = (e^t - 1, \sin t), \quad t = 20$

16. $c(t) = (\ln t, 3t^2 - t), \quad P = (0, 2)$

17. CAS Find the point on the cycloid $c(t) = (t - \sin t, 1 - \cos t)$ where the tangent line has slope $\frac{1}{2}$.

18. Find the points on $(t + \sin t, t - 2\sin t)$ where the tangent is vertical or horizontal.

19. Find the equation of the Bézier curve with control points

$$P_0 = (-1, -1), \quad P_1 = (-1, 1), \quad P_2 = (1, 1), \quad P_3(1, -1)$$

20. Find the speed at $t = \frac{\pi}{4}$ of a particle whose position at time t seconds is $c(t) = (\sin 4t, \cos 3t)$.

21. Find the speed (as a function of t) of a particle whose position at time t seconds is $c(t) = (\sin t + t, \cos t + t)$. What is the particle's maximal speed?

22. Find the length of $(3e^t - 3, 4e^t + 7)$ for $0 \le t \le 1$.

In Exercises 23 and 24, let $c(t) = (e^{-t}\cos t, e^{-t}\sin t)$.

23. Show that $c(t)$ for $0 \le t < \infty$ has finite length and calculate its value.

24. Find the first positive value of t_0 such that the tangent line to $c(t_0)$ is vertical, and calculate the speed at $t = t_0$.

25. CAS Plot $c(t) = (\sin 2t, 2\cos t)$ for $0 \le t \le \pi$. Express the length of the curve as a definite integral, and approximate it using a computer algebra system.

26. Convert the points $(x, y) = (1, -3), (3, -1)$ from rectangular to polar coordinates.

27. Convert the points $(r, \theta) = \left(1, \frac{\pi}{6}\right), \left(3, \frac{5\pi}{4}\right)$ from polar to rectangular coordinates.

28. Write $(x + y)^2 = xy + 6$ as an equation in polar coordinates.

29. Write $r = \dfrac{2\cos\theta}{\cos\theta - \sin\theta}$ as an equation in rectangular coordinates.

30. Show that $r = \dfrac{4}{7\cos\theta - \sin\theta}$ is the polar equation of a line.

31. GU Convert the equation

$$9(x^2 + y^2) = (x^2 + y^2 - 2y)^2$$

to polar coordinates, and plot it with a graphing utility.

32. Calculate the area of the circle $r = 3\sin\theta$ bounded by the rays $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

33. Calculate the area of one petal of $r = \sin 4\theta$ (see Figure 1).

34. The equation $r = \sin(n\theta)$, where $n \ge 2$ is even, is a "rose" of $2n$ petals (Figure 1). Compute the total area of the flower, and show that it does not depend on n.

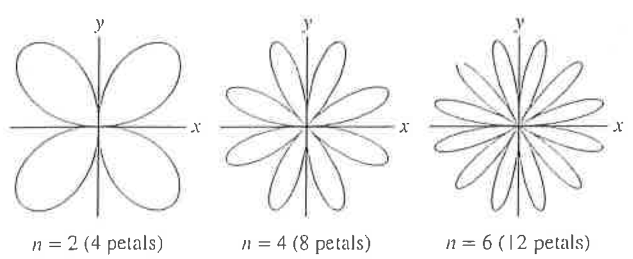


FIGURE 1 Plot of $r = \sin(n\theta)$.

35. Calculate the total area enclosed by the curve $r^2 = \cos\theta\, e^{\sin\theta}$ (Figure 2).

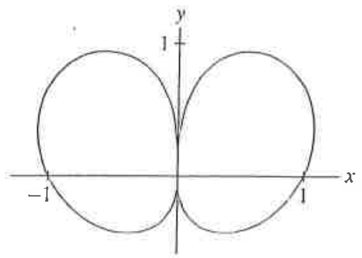


FIGURE 2 Graph of $r^2 = \cos\theta e^{\sin\theta}$.

36. Find the shaded area in Figure 3.

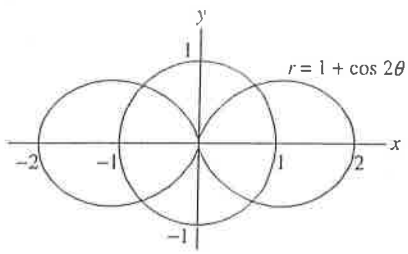


FIGURE 3

37. Find the area enclosed by the cardioid $r = a(1 + \cos\theta)$, where $a > 0$.

38. Calculate the length of the curve with polar equation $r = \theta$ in Figure 4.

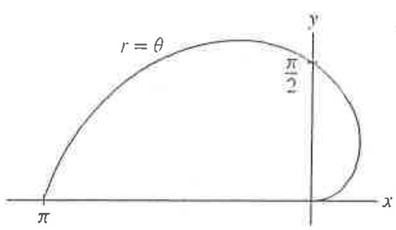


FIGURE 4

In Exercises 39–44, let $\mathbf{v} = \langle -2, 5\rangle$ *and* $\mathbf{w} = \langle 3, -2\rangle$.

39. Calculate $5\mathbf{w} - 3\mathbf{v}$ and $5\mathbf{v} - 3\mathbf{w}$.

40. Sketch $\mathbf{v}$, $\mathbf{w}$, and $2\mathbf{v} - 3\mathbf{w}$.

41. Find the unit vector in the direction of $\mathbf{v}$.

42. Find the length of $\mathbf{v} + \mathbf{w}$.

43. Express $\mathbf{i}$ as a linear combination $r\mathbf{v} + s\mathbf{w}$.

44. Find a scalar α such that $\|\mathbf{v} + \alpha\mathbf{w}\| = 6$.

45. If $P = (1, 4)$ and $Q = (-3, 5)$, what are the components of $\overrightarrow{PQ}$? What is the length of $\overrightarrow{PQ}$?

46. Let $A = (2, -1)$, $B = (1, 4)$, and $P = (2, 3)$. Find the point Q such that $\overrightarrow{PQ}$ is equivalent to $\overrightarrow{AB}$. Sketch $\overrightarrow{PQ}$ and $\overrightarrow{AB}$.

47. Find the vector with length 3 making an angle of $\frac{7\pi}{4}$ with the positive x-axis.

48. Calculate $3(\mathbf{i} - 2\mathbf{j}) - 6(\mathbf{i} + 6\mathbf{j})$.

49. Find the value of β for which $\mathbf{w} = \langle -2, \beta\rangle$ is parallel to $\mathbf{v} = \langle 4, -3\rangle$.

50. Let $\mathbf{r}_1(t) = \mathbf{v}_1 + t\mathbf{w}_1$ and $\mathbf{r}_2(t) = \mathbf{v}_2 + t\mathbf{w}_2$ be parametrizations of lines $\mathcal{L}_1$ and $\mathcal{L}_2$. For each statement (a)–(e), provide a proof if the statement is true and a counterexample if it is false.

(a) If $\mathcal{L}_1 = \mathcal{L}_2$, then $\mathbf{v}_1 = \mathbf{v}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

(b) If $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$, then $\mathbf{w}_1 = \mathbf{w}_2$.

(c) If $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$, then $\mathbf{v}_1 = \mathbf{v}_2$.

(d) If $\mathcal{L}_1$ is parallel to $\mathcal{L}_2$, then $\mathbf{w}_1 = \mathbf{w}_2$.

(e) If $\mathcal{L}_1$ is parallel to $\mathcal{L}_2$, then $\mathbf{w}_1 = \lambda\mathbf{w}_2$ for some scalar λ.

51. Sketch the vector sum $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$ for the vectors in Figure 5(A).

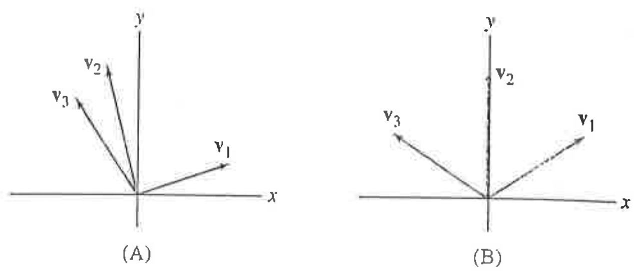


FIGURE 5

52. Sketch the sums $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_1 + 2\mathbf{v}_2$, and $\mathbf{v}_2 - \mathbf{v}_3$ for the vectors in Figure 5(B).

53. Use vectors to prove that the line connecting the midpoints of two sides of a triangle is parallel to the third side.

54. Calculate the magnitude of the forces on the two ropes in Figure 6.

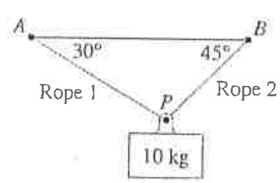


FIGURE 6

55. A 50-kg wagon is pulled to the right by a force $\mathbf{F}_1$ making an angle of 30° with the ground. At the same time the wagon is pulled to the left by a horizontal force $\mathbf{F}_2$.

(a) Find the magnitude of $\mathbf{F}_1$ in terms of the magnitude of $\mathbf{F}_2$ if the wagon does not move.

(b) What is the maximal magnitude of $\mathbf{F}_1$ that can be applied to the wagon without lifting it?

56. Find the angle between $\mathbf{v}$ and $\mathbf{w}$ if $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| = \|\mathbf{w}\|$.

57. Find $\|\mathbf{e} - 4\mathbf{f}\|$, assuming that $\mathbf{e}$ and $\mathbf{f}$ are unit vectors such that $\|\mathbf{e} + \mathbf{f}\| = \sqrt{3}$.

58. Find the area of the parallelogram spanned by vectors $\mathbf{v}$ and $\mathbf{w}$ such that $\|\mathbf{v}\| = \|\mathbf{w}\| = 2$ and $\mathbf{v} \cdot \mathbf{w} = 1$.

In Exercises 59–64, calculate the derivative indicated.

59. $\mathbf{r}'(t)$, $\quad \mathbf{r}(t) = \langle 1 - t, t^{-2} \rangle$

60. $\mathbf{r}'''(t)$, $\quad \mathbf{r}(t) = \langle t^3, 4t^2 \rangle$

61. $\mathbf{r}'(0)$, $\quad \mathbf{r}(t) = \langle e^{2t}, e^{-4t^2} \rangle$

62. $\mathbf{r}''(-3)$, $\quad \mathbf{r}(t) = \langle t^{-2}, (t+1)^{-1} \rangle$

63. $\dfrac{d}{dt} e^t \langle 1, t \rangle$

64. $\dfrac{d}{d\theta} \mathbf{r}(\cos\theta)$, $\quad \mathbf{r}(s) = \langle s, 2s \rangle$

In Exercises 65 and 66, calculate the derivative at $t = 3$, assuming that

$$\mathbf{r}_1(3) = \langle 1, 1 \rangle, \qquad \mathbf{r}_2(3) = \langle 1, 1 \rangle$$
$$\mathbf{r}_1'(3) = \langle 0, 0 \rangle, \qquad \mathbf{r}_2'(3) = \langle 0, 2 \rangle$$

65. $\dfrac{d}{dt}(6\mathbf{r}_1(t) - 4 \cdot \mathbf{r}_2(t))$ $\qquad$ 66. $\dfrac{d}{dt}\left(e^t \mathbf{r}_2(t)\right)$

67. Calculate $\displaystyle\int_0^3 \langle 4t + 3, t^2 \rangle \, dt$.

68. Calculate $\displaystyle\int_0^{\pi} \langle \sin\theta, \theta \rangle \, d\theta$.

69. A particle located at (1, 1) at time $t = 0$ follows a path whose velocity vector is $\mathbf{v}(t) = \langle 1, t \rangle$. Find the particle's location at $t = 2$.

70. Find the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ in $\mathbf{R}^2$ satisfying $\mathbf{r}'(t) = -\mathbf{r}(t)$ with initial conditions $\mathbf{r}(0) = \langle 1, 2 \rangle$.

71. Calculate $\mathbf{r}(t)$ assuming that

$$\mathbf{r}''(t) = \left\langle 4 - 16t, 12t^2 - t \right\rangle, \qquad \mathbf{r}'(0) = \langle 1, 0 \rangle, \qquad \mathbf{r}(0) = \langle 0, 1 \rangle$$

72. Solve $\mathbf{r}''(t) = \left\langle t^2 - 1, t + 1 \right\rangle$ subject to the initial conditions $\mathbf{r}(0) = \langle 1, 0 \rangle$ and $\mathbf{r}'(0) = \langle -1, 1 \rangle$.

73. A projectile fired at an angle of 60° lands 400 m away. What was its initial speed?

74. A force $\mathbf{F} = \langle 12t + 4, 8 - 24t \rangle$ (in newtons) acts on a 2-kg mass. Find the position of the mass at $t = 2$ s if it is located at (4, 6) at $t = 0$ and has initial velocity $\langle 2, 3 \rangle$ in m/s.

75. Find the unit tangent vector to $\mathbf{r}(t) = \langle \sin t, t \rangle$ at $t = \pi$.

Odd Answers!

Chapter 11 Review

1. (a), (c)

3. $c(t) = (1 + 2\cos t, 1 + 2\sin t)$. The intersection points with the y-axis are $\left(0, 1 \pm \sqrt{3}\right)$. The intersection points with the x-axis are $\left(1 \pm \sqrt{3}, 0\right)$.

5. $c(\theta) = (\cos(\theta + \pi), \sin(\theta + \pi))$ $\quad$ 7. $c(t) = (1 + 2t, 3 + 4t)$

9. $y = -\frac{x}{4} + \frac{37}{4}$ $\quad$ 11. $y = \frac{8}{(3-x)^2} + \frac{3-x}{2}$

13. $\left.\frac{dy}{dx}\right|_{t=3} = \frac{3}{14}$ $\quad$ 15. $\left.\frac{dy}{dx}\right|_{t=0} = \frac{\cos 20}{e^{20}}$

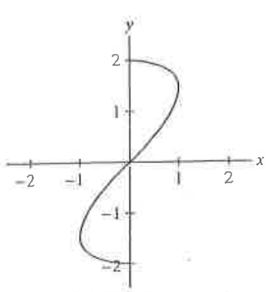
17. $(0, 1)$, $(\pi, 2)$, $(0.13, 0.40)$, and $(1.41, 1.60)$

19. $x(t) = -2t^3 + 4t^2 - 1$, $y(t) = 2t^3 - 8t^2 + 6t - 1$

21. $\frac{ds}{dt} = \sqrt{3 + 2(\cos t - \sin t)}$; maximal speed: $\sqrt{3 + 2\sqrt{2}}$

23. $s = \sqrt{2}$

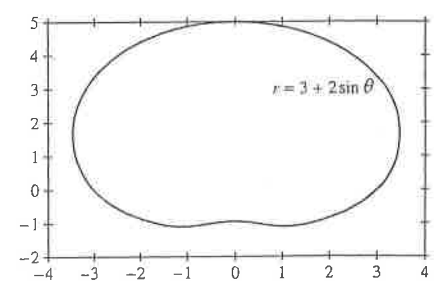
25.



$s = 2\int_0^{\pi} \sqrt{\cos^2 2t + \sin^2 t} \, dt \approx 6.0972$

27. $\left(1, \frac{\pi}{6}\right)$ and $\left(3, \frac{5\pi}{4}\right)$ have rectangular coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$.

29. $\sqrt{x^2 + y^2} = \frac{2x}{x-y}$ $\quad$ 31. $r = 3 + 2\sin\theta$



33. $A = \frac{\pi}{16}$ $\quad$ 35. $e - \frac{1}{e}$

Note: One needs to double the integral from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ in order to account for both sides of the graph.

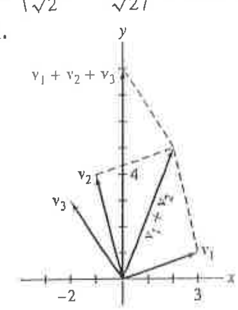
37. $A = \frac{3\pi a^2}{2}$

39. $\langle 21, -25 \rangle$ and $\langle -19, 31 \rangle$ $\quad$ 41. $\left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$

43. $\mathbf{i} = \frac{2}{11}\mathbf{v} + \frac{5}{11}\mathbf{w}$ $\quad$ 45. $\overrightarrow{PQ} = \langle -4, 1 \rangle$; $\left\|\overrightarrow{PQ}\right\| = \sqrt{17}$

47. $\left\langle \frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}} \right\rangle$ $\quad$ 49. $\beta = \frac{3}{2}$

51.



55. $\|\mathbf{F}_1\| = \frac{2\|\mathbf{F}_2\|}{\sqrt{3}}$; $\|\mathbf{F}_1\| = 980$ N

57. $\|\mathbf{e} - 4\mathbf{f}\| = \sqrt{13}$

59. $\mathbf{r}'(t) = \left\langle -1, -2t^{-3} \right\rangle$ 61. $\mathbf{r}'(0) = \langle 2, 0 \rangle$

63. $\frac{d}{dt} e^t \langle 1, t \rangle = e^t \langle 1, 1+t \rangle$

65. $\frac{d}{dt} (6\mathbf{r}_1(t) - 4\mathbf{r}_2(t))|_{t=3} = \langle 0, -8 \rangle$

67. $\int_0^3 \left\langle 4t + 3, t^2 \right\rangle \, dt = \langle 27, 9 \rangle$

69. $(3, 3)$ 71. $\mathbf{r}(t) = \left\langle 2t^2 - \frac{8}{3}t^3 + t, t^4 - \frac{1}{6}t^3 + 1 \right\rangle$

73. $v_0 \approx 67.279$ m/s

75. $\mathbf{T}(\pi) = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$